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Quantum groups $SO_q(N)$, $Sp_q(n)$ have q-determinants, too

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Abstract. We construct the q-deformed analogue of the completely antisymmetric tensors and the corresponding q-determinants $\det_q T$ for the quantum groups $SO_q(N)$, $Sp_q(n)$. The construction is based on the existence of the volume form in the algebra of exterior forms on the corresponding quantum spaces. We show that $\det_q T$ is central in $Fun(SO_q(N))$ (respectively $Fun(Sp_q(n))$) is group-like under the Hopf algebra comultiplication, and that its square is 1.

In [1], one-parameter deformations of the classical simple Lie groups and Lie algebras are presented. For each Lie group G a family $Fun(G_q)$ of Hopf algebras parameterized by $q \in \mathbb{C}$ ($q \equiv$ the parameter of deformation) is given, and for $q = 1$ (which corresponds to the so-called classical limit) $Fun(G_q)$ reduces to the Hopf algebra $Fun(G)$ of functions on G. With a suggestive expression, $Fun(G_q)$ is said to be the Hopf algebra of functions on the 'quantum group' G_q [2].

To build the Hopf algebra $Fun(SL_q(N))$ one chooses an R-matrix which allows the definition of the q-determinant $\det_q T$ of the generators T^i_j of $Fun(GL_q(N))$ as the only non-trivial central element in this algebra; then one can set $\det_q T = 1$ as the characterizing condition for $Fun(SL_q(N))$. On the contrary, the other Hopf algebras corresponding to simple groups of the classical series, namely $Fun(SO_q(N))$, $Fun(Sp_q(n))$, are characterized by quadratic relations in the generators T^i_j (the orthogonality relations with respect to a q-deformed metric, see equation (1) below), which guarantee that the corresponding transformations leave the 'distance' in the underlying quantum spaces unchanged. When $q = 1$ (i.e. in the so-called classical limit) for these relations it follows $\det(T)^2 = 1$. When $q \neq 1$, *a priori* it is not clear whether a determinant can be defined and whether its square is automatically one. From the definition of these quantum groups it turns out that this is the case. In the proof of this result we use the properties of the Euclidean (respectively symplectic) quantum spaces and of the corresponding algebras of differential forms on them; these prove to be very helpful tools. As intermediate interesting results we show the existence of a (unique) volume form on these quantum spaces, hence of the q-deformed analogue of the completely antisymmetric tensor ϵ, and we find a very simple commutation relation between the volume form and the coordinates. From the latter result it immediately follows that $\det_q T$ is central in $Fun(G_q)$. Then we prove that $(\det_q T)^2 = 1$. This is compatible with setting $\det_q(T) = 1$. Hence the mentioned volume form transforms as a scalar under the quantum group coaction, namely it has the expected transformation law we would require to an integration measure; in fact in [4] we have proved that it is possible, in the Euclidean case, to define an integration where the volume form play such a role.

The elements of the Hopf algebra $Fun(G_q)$, where G_q denotes one of the quantum groups $SO_q(2n+1)$, $Sp_q(n)$, $SO_q(2n)$ (quantum deformations of the classical series

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$\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{D}_n$, respectively) are formal ordered power series in the generating elements $\{T^i_j\}$; here $i, j = -n, -n+1, \ldots, -1, 0, 1, 2, \ldots, n$ for the first series and $i, j = -n, -n+1, \ldots, -1, 1, 2, \ldots, n$ for the last two. The elements T^i_j satisfy the relations

$$T^i_j C^{jl} T^k_l = \mathbf{1}_{Fun(G_q)} C^{ik} \qquad T^j_i C_{jl} T^l_k = \mathbf{1}_{Fun(G_q)} C_{ik} \tag{1}$$

and

$$\hat{R}^{ij}_{hk} T^h_l T^k_m = T^i_h T^j_k \hat{R}^{hk}_{lm}. \tag{2}$$

Here $C := \|C_{ij}\|$ denotes the q-deformed metric matrix, $\mathbf{1}_{Fun(G_q)}$ denotes the unit of the algebra $Fun(G_q)$. The former is explicitly given by

$$C_{ij} := \epsilon_i q^{-\rho_i} \delta_{i,-j} \tag{3}$$

where

$$(\rho_i) := \begin{cases} (n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2} \ldots, \frac{1}{2}-n) & \text{if } G_q = SO_q(2n+1) \\ (n, n-1, \ldots, 1, -1, \ldots, -n) & \text{if } G_q = Sp_q(n) \\ (n-1, n-2, \ldots, 0, 0, \ldots, 1-n) & \text{if } G_q = SO_q(2n) \end{cases} \tag{4}$$

and $\epsilon_i = 1$ if $G_q = SO_q(2n)$, $SO_q(2n+1)$, $\epsilon_i - \mathrm{sign}(i)$ if $G_q = Sp_q(2n)$. In the sequel N will denote the number $2n+1$ for the series $\mathcal{B}_n$ and the number $2n$ for the series $\mathcal{C}_n$, $\mathcal{D}_n$. $C^{-1} = C$, for the series $\mathcal{B}_n$, $\mathcal{D}_n$, whereas $C^{-1} = -C$ for the series $\mathcal{C}_n$; C^{ij} will denote $(C^{-1})_{ij}$. $\hat{R} := \|\hat{R}^{ij}_{hk}\|$ is the braid matrix and satisfies the Yang–Baxter equation (in the braid version)

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{5}$$

$\hat{R}$ is explicitly given by

$$\hat{R} = q\sum_{i\neq -i} e^i_i \otimes e^i_i + \sum_{\substack{i\neq j,-j \\ \text{or } i=j=0}} e^j_i \otimes e^i_j + q^{-1}\sum_{i\neq -i} e^{-i}_i \otimes e^i_{-i} + (q-q^{-1}) \times \left[\sum_{i<j} e^i_i \otimes e^j_j - \sum_{i<j} \epsilon_i\epsilon_j q^{-\rho_i+\rho_j} e^{-j}_i \otimes e^j_{-i}\right] \tag{6}$$

where $(e^i_j)^h_k := \delta^{ih}\delta_{jk}$. When $q = 1$ $\hat{R}$ reduces to the permutation matrix P, $P^{ij}_{hk} := \delta^i_k\delta^j_h$. The R-matrix defined by $R := P\hat{R}$ is lower triangular, i.e. $R^{ij}_{hk} = 0$ if either $i < h$, or $i = h$ and $j < k$. When $N > 2$ and $(1+q^2)(1+\epsilon q^{\epsilon-N+1})(1-\epsilon q^{\epsilon-N-1}) \neq 0$ the matrix $\hat{R}$ has the spectral decomposition

$$\hat{R} = q\mathcal{P}^{(+)} - q^{-1}\mathcal{P}^{(-)} + \epsilon q^{1-N}\mathcal{P}^0 \tag{7}$$

here $\epsilon = 1$ for $G_q = SO_q(N)$ and $\epsilon = -1$ for $G_q = Sp_q(n)$. When $G_q = SO_q(N)$ the orthogonal projectors $\mathcal{P}^{(+)}$, $\mathcal{P}^{(-)}\mathcal{P}^{(0)}$ have dimensions $[N(N+1)/2]-1$, $N(N-1)/2$, 1, and for $q = 1$ they reduce to the projectors over the irreducible corepresentations (symmetric modulo singlet, antisymmetric and singlet, respectively) of the tensor product $x \otimes x$ of the fundamental corepresentation (x) of $SO(N)$. When $G_q = Sp_q(n)$ they have dimensions $N(N+1)/2, [N(N-1)/2]-1, 1$. The matrix elements of $\mathcal{P}^{(0)}$ are given by $\mathcal{P}^{(0)ij}_{hk} = C^{ij}C_{hk}/Q_N$, where $Q_N = (1+q^{2-N})(q^N-1)/(q^2-1)$ for $G_q = SO_q(N)$ and $Q_N = 1-q^{-N/2}(q^{N+1}-1)/(q-1)$ for $G_q = Sp_q(n)$. Using the decomposition of the identity matrix $\mathbf{1} = \mathcal{P}^{(+)} + \mathcal{P}^{(-)} + \mathcal{P}^{(0)}$ we can invert relation (7) to obtain

$$\begin{aligned}\mathcal{P}^{(-)} &= \frac{1}{q+q^{-1}}[-\hat{R} + q\mathbf{1} + (q^{\epsilon-N} - q)\mathcal{P}^{(0)}] \\ \mathcal{P}^{(+)} &= \frac{1}{q+q^{-1}}[\hat{R} + q^{-1}\mathbf{1} - (q^{-1} + \epsilon q^{\epsilon-N})\mathcal{P}^{(0)}].\end{aligned} \tag{8}$$

$\hat{R}$ satisfies the equations

$$f(\hat{R})(T\otimes T) = (T\otimes T)f(\hat{R}) \tag{9}$$

$$f(\hat{R}_{12})\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}f(\hat{R}_{23}) \tag{10}$$

for any polynomial $f(t)$ in one variable t. We recall that the coproduct Δ is defined by

$$\Delta(\mathbf{1}_{Fun(G_q)}) = \mathbf{1}_{Fun(G_q)} \bigotimes \mathbf{1}_{Fun(G_q)} \qquad \Delta(T^i_j) = T^i_k \bigotimes T^k_j \tag{11}$$

on the basic variables T^i_j and on the unit $\mathbf{1}_{Fun(G_q)}$ of $Fun(G_q)$, and it is extended as a linear homomorphism to all $Fun(G_q)$:

$$\Delta(ab) = \Delta(a)\Delta(b) \qquad \Delta(a+b) = \Delta(a) + \Delta(b) \qquad \forall a, b \in Fun(G_q). \tag{12}$$

According to [1], a (left) $Fun(G_q)$ comodule C^N_{f,G_q} can be defined as the Poincaré algebra of power series in variables $\{y^i\}$, $(i = -n, 1-n, \ldots, n)$ satisfying relations

$$[f(\hat{R})]^{ij}_{hk}y^h y^k = 0 \tag{13}$$

where $f(t)$ is a polynomial function in one variable t. We recall that the (left) coaction δ of $Fun(G_q)$ on the comodule is defined by

$$\delta(\mathbf{1}_{C^N_{f,G_q}}) = \mathbf{1}_{Fun(G_q)} \bigotimes \mathbf{1}_{C^N_{f,G_q}} \qquad \delta(y^i) = T^i_k \bigotimes y^k \tag{14}$$

on the basic variables y^i and the unit $\mathbf{1}_{C^N_{f,G_q}}$ of C^N_{f,G_q}, and it is extended as a linear homomorphism to all C^N_{f,G_q}:

$$\delta(ab) = \delta(a)\delta(b) \qquad \delta(a+b) = \delta(a) + \delta(b) \qquad \forall a, b, \in C^N_{f,G_q}. \tag{15}$$

For instance, in [1] the algebra O^N_q of the functions on the N-dimensional quantum Euclidean space was defined by the $N(N-1)/2$ independent conditions

$$\mathcal{P}^{(-)ij}_{hk}x^h x^k = 0 \tag{16}$$

whereas the algebra Sp^N_q $(N = 2n)$ of functions of the quantum symplectic space was defined by the $N(N-1)/2$ independent conditions

$$\mathcal{P}^{(-)ij}_{hk}x^h x^k = 0 \qquad \mathcal{P}^{(0)ij}_{hk}x^h x^k = 0 \Leftrightarrow (\hat{R} - q\mathbf{1})^{ij}_{hk}x^h x^k = 0. \tag{17}$$

Similarly, the algebra $\Lambda_q O^N$ of exterior forms on the N-dimensional quantum Euclidean space was defined by the conditions

$$\mathcal{P}^{(+)ij}_{\ \ hk}\xi^h\xi^k = 0. \tag{18}$$

Since $\xi C\xi := \xi^i C^{ij}\xi^j$ is central, we are free to add to the $[N(N+1)/2]-1$ independent relations (18) one more relation by setting

$$\mathcal{P}^{(0)ij}_{\ \ hk}\xi^h\xi^k = 0 \Leftrightarrow \xi C\xi = 0. \tag{19}$$

We assume this more stringent definition so as to recover the ordinary algebra of exterior forms in the classical limit $q=1$. In fact, when $q=1$ $(\mathcal{P}^{(+)}+\mathcal{P}^{(0)})^{ij}_{hk} = \frac{1}{2}(\delta^i_h\delta^j_k + \delta^i_k\delta^j_h)$, therefore equations (18) and (19) imply the usual commutation relations $\xi^i\xi^j + \xi^j\xi^i = 0$. Incidentally, we note that the pair of conditions (18) and (19) comes out in a natural way by the construction of the differential forms ξ^is as exterior derivatives $\mathrm{d}x^i$s of the coordinates x^is, where the latter satisfy condition (16) (the exterior derivative d is required to satisfy linearity, nilpotency, and the Leibnitz rule) [3].

In a similar way, we define the algebra $\Lambda_q Sp^N$ ($N=2n$) of exterior forms on the N-dimensional quantum symplectic space by the conditions (18) alone (remember that in the symplectic case they amount to $N(N+1)/2$ independent relations), which reduce to the classical commutation relations $\xi^i\xi^j + \xi^j\xi^i = 0$ for $q=1$.

We are going to show that the pair of conditions (18) and (19) (respectively the condition (19)) is enough to order the variables ξ^is belonging to $\Lambda_q O^N$ (respectively $\Lambda_q Sp^N$) in any prescribed way, so that we can uniquely define the volume N-form and therefore the q-deformed analogue $\epsilon_q^{i_1 i_2 \ldots i_N}$ of the completely antisymmetric tensor. To this end we write the pair of conditions (18) and (19) (respectively condition (19)) in a more manageable form. Note that in the case $G_q = SO_q(N)$ (in view of the second of formulae (8)) the pair (18) and (19) is equivalent to

$$F^{ij}_{hk}\xi^h\xi^k = 0 \qquad F = \hat{R} + q^{-1}\mathbf{1}. \tag{20}$$

A complete set of independent relations can be obtained from (20) (respectively (18) alone) by choosing either $i \leqslant j$, $i \neq -j$ or $-i = j \geqslant 0$. By combining them linearly we can put them in the following more explicit form. The equations defining the quantum Euclidean (respectively symplectic) space amount to the system

$$q\xi^i\xi^j + \xi^j\xi^i = 0 \qquad i<j \qquad i \neq -j \tag{21a}$$

$$\xi^i\xi^i = 0 \qquad i \neq 0 \tag{21b}$$

$$\xi^l\xi^{-l}q^{-1} + \xi^{-l}\xi^l q = \xi^{l-1}\xi^{1-l} + \xi^{1-l}\xi^{l-1} \qquad n \geqslant l > 1 \tag{21c}$$

$$q\xi^{-1}\xi^1 + q^{-1}\xi^1\xi^{-1} = \begin{cases} (q-q^{-1})\sum\limits_{i\leqslant -1} q^{-\rho_i}\xi^i\xi^{-i} & \text{if } G_q = SO_q(2n) \\ 0 & \text{if } G_q = Sp_q(2n) \end{cases} \tag{21d}$$

$$\left\{\begin{array}{ll} q^{-3/2}(\xi^{-1}\xi^1 + \xi^1\xi^{-1}) - (q-q^{-1})\sum\limits_{i<-1} q^{-\rho_i}\xi^i\xi^{-i} = 0 & (21e) \\ & \text{if } G_q = SO_q(2n+1). \\ (q^{1/2}+q^{-1/2})\xi^0\xi^0 = q\xi^{-1}\xi^1 + q^{-1}\xi^1\xi^{-1} & (21f) \end{array}\right.$$

From these relations we realize that any product $\xi^l\xi^m$ can be written as a combination of terms of the type $\xi^i\xi^j$, where i,j satisfy a prescribed order relation. In particular, if we take the usual order relation, i.e. we require $i<j$ for all i,j, we see that this statement is true since:

(i) Equations (21a) and (21b) let us order $\xi^l\xi^m$ if $l \neq -m$.

(ii) If $l = -m$ (say $l \geqslant 0$), then:

(a) If $l > 1$ repeated application of equation (21c) let us write $\xi^l\xi^{-l}$ as a combination of $\xi^j\xi^{-j}$, $-n \leqslant j \leqslant 1$: it remains to order $\xi^1\xi^{-1}$, $\xi^0\xi^0$.

(b) If $l = 1$ and $G_q = SO_q(2n)$ (respectively $G_q = Sp_q(n)$, $G_q = SO_q(2n+1)$) we use equation (21d) (respectively (21e) and (21f)) to write $\xi^1\xi^{-1}$ as a combination of $\xi^j\xi^{-j}$, $-n \leqslant j \leqslant -1$.

(c) If $l = 0$ (and $G_q = SO_q(2n+1)$) equation (21g) and the remarks of point (b) let us reduce $\xi^0\xi^0$ to a combination of $\xi^j\xi^{-j}$ with $-n \leqslant j \leqslant -1$.

By repeated application of the ordering procedure we can rewrite any monomial $M := \xi^{i_1}\xi^{i_2}\ldots\xi^{i_k}$ as a combination of monomials $\xi^{j_1}\xi^{j_2}\ldots\xi^{j_k}$, where the indices $j_1, j_2, \ldots, j_k$ satisfy a prescribed order relation, in particular the usual one $j_1 < j_2 < \ldots < j_k$. It is realized immediately that if $k = N$, just as when $q = 1$, any monomial M is either zero or proportional to

$$\mathrm{d}V := \begin{cases} \xi^{-n}\xi^{-n+1}\ldots\xi^{-1}\xi^{1}\ldots\xi^{n-1}\xi^{n} & \text{if } N \text{ is even} \\ \xi^{-n}\xi^{-n+1}\ldots\xi^{-1}\xi^{0}\xi^{1}\ldots\xi^{n-1}\xi^{n} & \text{if } N \text{ is odd} \end{cases} \tag{22}$$

which will therefore be called the 'volume form'; whereas if $k \geqslant N+1$ $M = 0$. We define the q-deformed antisymmetric tensor $\epsilon_q^{i_1 i_2 \ldots i_N}$ by

$$\xi^{i_1}\xi^{i_2}\ldots\xi^{i_N} := \epsilon_q^{i_1 i_2\ldots i_N}\mathrm{d}V. \tag{23}$$

It satisfies the relations

$$\mathcal{P}^{(+)lm}_{i_j i_{j+1}}\epsilon_q^{i_1 i_2\ldots i_N} = 0 = \mathcal{P}^{(0)lm}_{i_j i_{j+1}}\epsilon_q^{i_1 i_2\ldots i_N} \qquad j = 1, 2, \ldots, N-1 \tag{24}$$

if $G_q = SO_q(N)$, and if $G_q = Sp_q(n)$ it only satisfies the first one of these relations. If N is even, then one easily realizes that $\epsilon_q^{i_1 i_2\ldots i_N}$ will be different from zero only if $(i_1, i_2, \ldots, i_N)$ is a permutation of $(-n, -n+1, \ldots, -1, 1, \ldots, n-1, n)$. If N is odd, $\epsilon_q^{i_1 i_2\ldots i_N}$ can be different from zero also if $n_0 > 1$ (n_0 odd) elements of the row $(i_1, i_2, \ldots, i_N)$ are equal to 0, and the remaining ones are all different from each other. A glance at the relation

$$\begin{aligned}(\xi^0)^{2h} &= \left(\sum_{j<0}\xi^j\xi^{-j}q^{-\rho_j}\frac{q-q^{-1}}{1+q^{-1}}\right)^h \\ &= \left(\frac{q-q^{-1}}{1+q^{-1}}\right)^h \sum_{j_1<j_2<\ldots<j_h<0} q^{h(h-1)-(\rho_{j_1}+\rho_{j_2}\ldots+\rho_{j_h})}h!\xi^{j_1}\ldots\xi^{j_h}\xi^{-j_h}\ldots\xi^{-j_1}\end{aligned} \tag{25}$$

$(0 < h \leqslant n)$ will convince the reader that this condition is not sufficient to guarantee that $\epsilon_q^{i_1 i_2\ldots i_N} \neq 0$. Consider for instance a monomial such as $\xi^{-n}\xi^{1-n}\ldots\xi^{-1}(\xi^0)^{2h+1}$; it vanishes, as in any term obtained upon use of relation (25) at least one ξ^{-j} with a $j > 0$ will appear two times. In fact, if ξ^j appears in a term in the RHS of equation (25), then in the same term there appears also ξ^{-j}. Therefore $\epsilon_q^{-n,1-n,\ldots,-1,0,0,\ldots} = 0$. To avoid unessential complications in the notation assume that $i_j \neq 0$ for $j \leqslant N - n_0$, $i_j = 0$ for $j > N - n_0$ and $n_0 = 2h+1$. Then a little reasoning will convince the reader that, more generally, $\epsilon_q^{i_1 i_2\ldots i_N}$ will be zero if the sets $Y := \{i_1, i_2, \ldots, i_{N-n_0}\}$, $Z := \{-i_1, -i_2, \ldots, -i_{N-n_0}\}$ satisfy the condition $Y \cup Z = \{-n, -n+1, \ldots, -1, 1, \ldots, n\}$.

Now we are in the condition to define the q-deformed determinant of $T = \|T^i_j\|$ in the usual way. By definition

$$\det_q T := T^{-n}_{i_1} T^{-n+1}_{i_2} \dots T^{n}_{i_N} \epsilon_q^{i_1 i_2 \dots i_N} \tag{26}$$

or, equivalently,

$$\delta(\mathrm{d}V) := \det_q T \bigotimes \mathrm{d}V. \tag{27}$$

Note that the property $\xi^{j_1}\xi^{j_2}\dots\xi^{j_N} \propto \mathrm{d}V$ implies $T^{j_1}_{i_1} T^{j_2}_{i_2} \dots T^{j_N}_{i_N} \epsilon_q^{i_1 i_2 \dots i_N} \propto \det_q T$. To find the result of the coaction Δ of G_q on $\det_q T$ we use the standard argument used in [1] for the quantum group $SU_q(N)$: applying both sides of the identity

$$(\Delta \bigotimes \mathbf{1}_\Lambda) \circ \delta = (\mathbf{1}_{Fun(G_q)} \bigotimes \delta) \circ \delta \tag{28}$$

to $\mathrm{d}V$ we find

$$\Delta(\det_q T) = \det_q T \bigotimes \det_q T \tag{29}$$

i.e. $\det_q T$ is group-like under the comultiplication Δ.

We give the explicit expression for the tensor ϵ_q and the q-determinant in the case $G_q = SO_q(3)$:

$$\begin{gathered}\epsilon_q^{-101} = 1 \qquad \epsilon_q^{-110} = -q \qquad \epsilon_q^{0-11} = -q \qquad \epsilon_q^{01-1} = q \qquad \epsilon_q^{10-1} = -q^2 \\ \epsilon_q^{1-10} = q \qquad \epsilon_q^{000} = -q(q^{1/2} - q^{-1/2}) \qquad \epsilon_q^{ijk} = 0 \qquad \text{otherwise.}\end{gathered} \tag{30}$$

The determinant is given by

$$\begin{aligned}\det_q T = {} & T^{-1}_{-1} T^0_0 T^1_1 - q T^{-1}_{-1} T^0_1 T^1_0 - q T^{-1}_0 T^0_{-1} T^1_1 + q T^{-1}_0 T^0_1 T^1_{-1} - q^2 T^{-1}_1 T^0_0 T^1_{-1} \\ & + q T^{-1}_1 T^0_{-1} T^1_0 - q(q^{1/2} - q^{-1/2}) T^{-1}_0 T^0_0 T^1_0.\end{aligned} \tag{31}$$

We see that $\det_q T$ is the sum of seven terms, one more (the one proportional to $T^{-1}_0 T^0_0 T^1_0$) than in the classical case.

Next, we show that $\det_q T$ is central in $Fun(G_q)$. The proof will be based on the fact that the $R = P\hat{R}$ matrix for any G_q is triangular and on the possibility of constructing the algebra $D\Lambda_q O^N$ (respectively $D\Lambda_q Sp^N$) of differential forms on the quantum space O^N_q (respectively Sp^N_q), i.e. the algebra of formal power series in the x^i, ξ^j variables. In other terms the latter algebra should contain both $\Lambda_q O^N$ (respectively $\Lambda_q Sp^N$) and O^N_q (respectively Sp^N_q) as subalgebras and should be a $Fun(SO_q(N))$ (respectively $Fun(Sp_q(n))$) comodule with respect to the coaction defined by equations (14) and (15) (where now the basic variables y^i can be either x^i or ξ^i, and a, b belong to $D\Lambda_q O^N$ (respectively $D\Lambda_q Sp^N$)). To perform this construction it is enough to prescribe commutation relations between the x^is and the ξ^js of the type

$$x^i \xi^j = M^{ij}_{hk} \xi^h x^k \tag{32}$$

in such a way that they are compatible with the defining relations (16), (18) and (19) (respectively (17) and (18)) of O_q^N, $\Lambda_q O^N$ (respectively $Sp_q(n)$, $\Lambda_q Sp_q^N$). In the $SO_q(N)$ case, since

$$(\mathcal{P}^{(-)ij}_{hk} x^h x^k)\xi^l = \mathcal{P}^{(-)ij}_{hk} M^{hr}_{uv} M^{kl}_{rs} \xi^u x^v x^s \tag{33}$$

and the LHS vanishes because of relation (16), to consistently impose the latter relation we have also to make the RHS vanish. This is automatically guaranteed by equation (10) (where we choose $f(\hat{R}) = \mathcal{P}^{(-)}$) if we take M proportional to either $\hat{R}$ or $\hat{R}^{-1}$. Similarly, such an M makes both sides of the equation

$$x^l(\mathcal{P}^{ij}_{hk}\xi^h\xi^k) = \mathcal{P}^{ij}_{hk} M^{lh}_{rs} M^{sk}_{uv} \xi^r \xi^u x^v \tag{34}$$

(where $\mathcal{P} = \mathcal{P}^{(+)}$, $\mathcal{P}^{(0)}$) vanish, if we impose relations (18) and (19). In the sequel we will take

$$M = q\hat{R}. \tag{35}$$

By repeated application of formula (32) we find the following commutation rule of x^i with $\mathrm{d}V$:

$$x^i \mathrm{d}V = q^N R^{-n\,i}_{l_1 v_1} R^{-n+1\,v_1}_{l_2 v_2} \ldots R^{n-1\,v_{N-2}}_{l_{N-1}\,v_{N-1}} R^{n\,v_{N-1}}_{l_N v_N} \epsilon_q^{l_1 l_2 \ldots l_N} \mathrm{d}V x^{v_N} \tag{36}$$

(here $R = P\hat{R}$). Because of the lower-triangularity of R, the matrix element $R^{-ni}_{l_1 v_1}$ is different from zero only when $l_1 = -n$, and a glance at the explicit expression for R shows that $R^{-ni}_{-nv_1} = \delta^i_{v_1} \cdot \alpha_{v_1}$, with $\alpha_{v_1} \neq 0$. Therefore in formula (36) we can replace l_1 by $-n$ and $R^{-ni}_{-nv_1}$ by $\delta^i_{v_1} \cdot \alpha_{v_1}$. Then the ϵ_q tensor forces l_2 to run over values $\geqslant -n+1$. Next, we use the triangularity of R to reduce $R^{-n+1 v_1}_{l_2 v_2}$ to $\delta^{v_1}_{v_2} \cdot \alpha_{v_2}$. Using the same kind of argument again and again we see that after n (respectively $n+1$) steps if N is even (respectively odd) relation (36) has become

$$x^i \mathrm{d}V = \alpha' R^{1\,i}_{l_{N-n+1} v_{N-n+1}} R^{2 v_{N-n+1}}_{l_{N-n+2} v_{N-n+2}} \ldots R^{n\,v_{N-1}}_{l_N v_N} \epsilon_q^{-n,1-n,\ldots,l_{N-n+1}\ldots l_N} \mathrm{d}V x^{v_N}. \tag{37}$$

Because of the remarks following equation (25), both for even and odd $N\epsilon_q^{-n,1-n,\ldots,l_{N-n+1}\ldots l_N}$ will be different from zero only if $(l_{N-n+1}, l_{N-n+2}, \ldots, l_N)$ is a permutation of $(1, 2, \ldots, n)$. Then the enforcement of the same kind of argument (based on the lower triangularity of R) as before will reduce relation (37) after n steps to

$$x^i \mathrm{d}V = \alpha \mathrm{d}V x^i \qquad \alpha \neq 0 \tag{38}$$

which is an interesting result by itself.

Now it is straightforward to show that $\det_q T$ is central in $Fun(G_q)$. Applying the coaction δ to both sides of equation (38) we obtain

$$T^i_j \det_q T \bigotimes x^j \mathrm{d}V = \alpha(\det_q T) T^i_j \bigotimes \mathrm{d}V x^j = (\det_q T) T^i_j \bigotimes x^j \mathrm{d}V \tag{39}$$

whence

$$T^i_j (\det_q T) = (\det_q T) T^i_j. \tag{40}$$

Formula (27) (together with equation (40)) shows that $\mathrm{d}V$ has the right transformations properties we would require to consider it as an integration measure. Actually in [4] we have shown that it is possible to introduce an integration on the quantum Euclidean space (based on Stoke's theorem) where $\mathrm{d}V$ plays the role of volume form.

Reasoning in a very similar way we can prove that $(\det_q T)^2 = 1$. This is the exact analogue of the classical situation. Let us consider the algebra and left $Fun(G_q)$ comodule generated by two sets of 1-forms $\{\xi^i\}$, $\{\xi'^j\}$ both satisfying conditions (18) and (19). As we did in the case of the algebras $D\Lambda_q O^N$, $D\Lambda_q Sp^n$, we have to impose commutation relations of the type

$$\xi^i \xi'^j = M^{ij}_{hk} \xi'^h \xi^k \tag{41}$$

with M proportional to $\hat{R}$ or $\hat{R}^{-1}$, for instance $M = -\hat{R}$. Now let us apply the coaction to the form $(\xi^i C_{ij} \xi'^j)^N$. Using relations (1) we obtain

$$\delta((\xi^i C_{ij} \xi'^j)^N) = \mathbf{1}_{Fun(G_q)} \bigotimes (\xi^i C_{ij} \xi'^j)^N. \tag{42}$$

On the other hand, using relations (41) to move all ξ' to the right of all the ξ we find that $(\xi^i C_{ij} \xi'^j)^N = \beta \mathrm{d}V \cdot \mathrm{d}V'$ with $\beta \neq 0$, and therefore

$$\delta((\xi^i C_{ij} \xi'^j)^N) = (\det_q T)^2 \bigotimes (\xi^i C_{ij} \xi'^j)^N. \tag{43}$$

Comparing equations (42) and (43) we derive the claimed equality $(\det_q T)^2 = 1$.

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